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STARLIKENESS OF SOME UNIVALENT FUNCTIONS

MANPREET KAUR [1], SUSHMA GUPTA [2] AND SUKHJIT SINGH [3]

Abstract. In the present paper we study a class $\mathcal{N}(\lambda)$ consisting of analytic functions f which satisfy the condition

$$\left|-z^3\left(\frac{z}{f(z)}\right)''' + f'(z)\left(\frac{z}{f(z)}\right)^2 - 1\right| \le \lambda,$$

where $0 < \lambda \le 1$ and $f(z)/z \ne 0$ in the open unit disk E. For given real numbers $\alpha, 0 < \alpha \le 1$, or $\gamma, 0 \le \gamma < 1$, we determine ranges of λ such that functions in $\mathcal{N}(\lambda)$ are strongly starlike of order α or starlike of order γ.

1. Introduction

Let $\mathcal{A}$ denote the class of all functions f that are analytic in the open unit disk $E = \{z : |z| < 1\}$ and are normalized by the conditions $f(0)=f'(0) - 1 = 0$. We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions univalent in E. A function $f \in \mathcal{A}$ is said to be starlike in E if f is univalent and $f(E)$ is starlike domain with respect to $z = 0$. The class of all starlike functions is denoted by $\mathcal{S}^*$. It is well known that $f \in \mathcal{A}$ is starlike with respect to the origin if and only if

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in E.$$

For $0 \le \gamma < 1$, we define the class $\mathcal{S}^*(\gamma)$ (called the class of starlike functions of order γ) as under:

$$\mathcal{S}^*(\gamma) = \left\{f \in \mathcal{A} : \text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \gamma, \quad z \in E\right\}.$$

Clearly, $\mathcal{S}^*(0) \equiv \mathcal{S}^*$. A function $f \in \mathcal{A}$ is said to be strongly starlike of order α, $0 < \alpha \le 1$ in E iff it satisfies $|\arg(zf'(z)/f(z))| < \alpha\pi/2,\ z \in E$. We denote by $\mathcal{S}_\alpha$ the class of all such functions. Note that $\mathcal{S}_1 \equiv \mathcal{S}^*(0) \equiv \mathcal{S}^*$. For $0 < \alpha < 1$, $\mathcal{S}_\alpha$ consists only of bounded starlike functions and therefore, the inclusion $\mathcal{S}_\alpha \subset \mathcal{S}^*$ is proper.
We say that $f \in R(\gamma), \gamma \in (0, 1]$ if $f \in \mathcal{A}$ and $|\arg f'(z)| < \gamma\pi/2$. It is well-known that the

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functions in $R(1)$ are Close-to-convex and hence univalent in E (see, [2, 14]).
For two functions f and g analytic in E with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, their convolution or Hadamard product is denoted by $f * g$ and is an analytic function in E defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

For a real number $\lambda, \lambda \geq 0$, and an analytic function f with $z/f(z) \neq 0$ in E define

$$\mathcal{U}(\lambda) = \left\{ f \in \mathcal{A} : \left| f'(z) \left(\frac{z}{f(z)} \right)^2 - 1 \right| \leq \lambda,\ z \in E \right\},$$
$$\mathcal{P}(\lambda) = \left\{ f \in \mathcal{A} : \left| \left(\frac{z}{f(z)} \right)'' \right| \leq \lambda,\ z \in E \right\},$$
$$\mathcal{M}(\lambda) = \left\{ f \in \mathcal{A} : \left| z^2 \left(\frac{z}{f(z)} \right)'' + f'(z) \left(\frac{z}{f(z)} \right)^2 - 1 \right| \leq \lambda,\ z \in E \right\},$$
$$\mathcal{N}(\lambda) = \left\{ f \in \mathcal{A} : \left| -z^3 \left(\frac{z}{f(z)} \right)''' + f'(z) \left(\frac{z}{f(z)} \right)^2 - 1 \right| \leq \lambda,\ z \in E \right\}.$$

Denote the classes $\mathcal{U}(1)\,, \mathcal{P}(1)\,, \mathcal{M}(1)$ and $\mathcal{N}(1)$ by $\mathcal{U}\,, \mathcal{P}\,, \mathcal{M}$ and $\mathcal{N}$, respectively.

The study of these classes started in 1972 when Ozaki and Nunokawa [9] proved that the functions in $\mathcal{U}(\lambda)$ are univalent for $0 < \lambda \leq 1$. In 2001, Obradović and Ponnusamy [4] proved that $\mathcal{P}(2\lambda) \subsetneq \mathcal{U}(\lambda)$. It is easy to observe that the Koebe function $z/(1-z)^2$ belongs to $\mathcal{U}$ but $\mathcal{U} \nsubseteq \mathcal{S}^*$ (see[1]). Ponnusamy and Vasundhra [12] obtained conditions on λ so that $\mathcal{U}(\lambda) \subseteq \mathcal{S}^*$. For further details on these classes including some interesting generalizations we refer to [3, 4, 8, 11, 12, 13].

In 2011, Obradović and Ponnusamy [5] studied the class $\mathcal{M}(\lambda)$ and further, in 2012 , in [6], they investigated the class $\mathcal{N}$ and proved the strict inclusions $\mathcal{N} \subsetneq \mathcal{M} \subsetneq \mathcal{P} \subsetneq \mathcal{U} \subsetneq \mathcal{S}$. They also obtained some necessary and sufficient conditions for functions to be in the class $\mathcal{N}$ and conjectured that $\mathcal{N}$ is not contained in $\mathcal{S}^*$. The work presented in this paper is inspired by this open problem. We study the class $\mathcal{N}(\lambda)$ where $0 < \lambda \leq 1$ and find conditions on the parameter λ so that the the functions in the class $\mathcal{N}(\lambda)$ are starlike or are strongly starlike of some order. We also find $\mathcal{N}$-radius of a function defined in terms of a function in the class $\mathcal{U}$. Note that for two subclasses $\mathcal{F}$ and $\mathcal{G}$ of $\mathcal{A}$, we say that r_0 is the $\mathcal{G}$-radius in $\mathcal{F}$ if for every $f \in \mathcal{F}, r^{-1} f(rz) \in \mathcal{G}$ for $r \leq r_0$ and r_0 is the maximum value for which this holds.

2. PRELIMINARY LEMMAS

Let $\mathcal{P}_n$ denote the class of functions p analytic in E such that $p^{(k)}(0) = 0$ for $k = 0, 1, 2.....n$, where $p^{(0)}(0)=p(0)$. Similarly with $w^{(0)}(0)=w(0)$, we set $\mathcal{B}_n=\{w$: w is analytic, $|w(z)| \leq 1$ in E and $w^{(k)}(0) = 0$ for $k = 0, 1, 2....n$ $\}$. For each $w(z) \in \mathcal{B}_k$, Schwarz's lemma gives: $|w(z)| \leq |z|^{k+1}$ for $n = 0, 1, 2, ...k$ and $z \in E$.
To prove our results, we shall need the following lemmas.

Lemma 2.1. [7, Lemma1c] *Let $f \in \mathcal{U}$ have the series representation $z/f(z) = 1+\sum_{n=1}^{\infty} b_n z^n$. Then*

$$\sum_{n=2}^{\infty}(n-1)^2|b_n|^2 \leq 1.$$

Lemma 2.2. [6, Theorem 2] *Let $\phi(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ be a non-vanishing analytic function in E and that it satisfies the condition*

$$\sum_{n=2}^{\infty}(n-1)^3|b_n| \leq 1.$$

Then the function $f(z) = z/\phi(z)$ is in $\mathcal{N}$.

3. MAIN RESULTS

In the following result we determine a range of λ such the $\mathcal{N}(\lambda) \subset \mathcal{S}_\alpha$ for some given real number $\alpha, 0 < \alpha \leq 1$.

Theorem 3.1. *Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ be in $\mathcal{N}(\lambda)$ and let $\alpha, 0 < \alpha \leq 1,$ be some real number and $|a_2| \leq \sin(\alpha\pi/2)$. Then $f \in \mathcal{S}_\alpha$ provided $0 < \lambda \leq \lambda_*(\alpha, |a_2|)$, where*

$$\lambda_*(\alpha, |a_2|) = \frac{-|a_2|\cos(\alpha\pi/4) + \sin(\alpha\pi/4)\sqrt{4\cos^2(\alpha\pi/4) - |a_2|^2}}{2\cos(\alpha\pi/4)}. \tag{3.1}$$

Proof. As $f \in \mathcal{N}(\lambda)$, so there exists a function $w \in \mathcal{B}_1$ such that

$$-z^3\left(\frac{z}{f(z)}\right)''' + f'(z)\left(\frac{z}{f(z)}\right)^2 - 1 = \lambda w(z). \tag{3.2}$$

Write

$$p(z) = f'(z)\left(\frac{z}{f(z)}\right)^2 - 1 = \left(\frac{z}{f(z)}\right) - z\left(\frac{z}{f(z)}\right)' - 1. \tag{3.3}$$

Obviously, p is analytic in E with $p(0) = p'(0) = 0$. Hence $p \in \mathcal{P}_1$. Then (3.2) is equivalent to

$$z^2 p''(z) - zp'(z) + p(z) = \lambda w(z). \tag{3.4}$$

Now, let

$$p(z) = \sum_{k=2}^{\infty} p_k z^k \quad \text{and} \quad w(z) = \sum_{k=2}^{\infty} w_k z^k.$$

A comparison of the coefficients of z^k on both sides in (3.4) gives that

$$p_k = \frac{\lambda w_k}{(k-1)^2} \quad \text{for} \quad k \geq 2.$$

Using this, we see that

$$p(z) = \lambda \sum_{k=2}^{\infty} \frac{w_k}{(k-1)^2} z^k = \lambda w(z) * \sum_{k=1}^{\infty} \frac{1}{k^2} z^{k+1}$$

Next, we recall (see [10]) that

$$\sum_{k=1}^{\infty} \frac{1}{(k+a)^q} z^k = \frac{1}{\Gamma(q)} \int_0^1 z(\log(1/t))^{q-1} \frac{t^a}{1-tz} dt \quad \text{for Re } a > -1 \text{ and Re } q > 1.$$

Using this result, it follows that

$$\begin{aligned} p(z) &= \lambda w(z) * z^2 \int_0^1 \frac{\log(1/t)}{1-tz} dt \\ &= \lambda \int_0^1 \frac{w(tz)}{t^2} \log(1/t) dt. \end{aligned} \tag{3.5}$$

As $w \in \mathcal{B}_1$, from Schwarz's Lemma it follows that $|w(tz)| \leq |(tz)|^2$ and, in view of (3.3) and (3.5), we find that

$$\left| f'(z) \left(\frac{z}{f(z)} \right)^2 - 1 \right| \leq \lambda \int_0^1 \frac{|w(tz)|}{t^2} \log(1/t) dt \leq \lambda |z|^2 < \lambda,$$

which gives

$$\left| \arg \left[f'(z) \left(\frac{z}{f(z)} \right)^2 \right] \right| \leq \arcsin(\lambda). \tag{3.6}$$

In view of (3.3), we can write (3.5) as

$$-z\left(\frac{z}{f(z)}-1+a_2z\right)'+\left(\frac{z}{f(z)}-1+a_2z\right)=\lambda\int_0^1\frac{w(tz)}{t^2}\log(1/t)dt.$$

From this we easily obtain

$$\frac{z}{f(z)}=1-a_2z-\frac{\lambda}{2}\int_0^1\frac{w(tz)}{t^2}(\log(1/t))^2dt. \tag{3.7}$$

Then, as $w\in\mathcal{B}_1$ and $z\in E$, we have

$$\begin{aligned}\left|\frac{z}{f(z)}-1\right| &\le |a_2||z|+\frac{\lambda}{2}\int_0^1\frac{|w(tz)|}{t^2}(\log(1/t))^2dt,\\ &\le |a_2||z|+\frac{\lambda}{2}|z|^2\int_0^1(\log(1/t))^2dt,\\ &\le |a_2||z|+\frac{\lambda}{2}\Gamma 3|z|^2,\\ &< |a_2|+\lambda\,.\end{aligned}$$

From the above inequality we conclude that

$$\left|\arg\left(\frac{z}{f(z)}\right)\right|\le\arcsin(|a_2|+\lambda). \tag{3.8}$$

Further

$$\arg\left(\frac{zf'(z)}{f(z)}\right)=\arg\left[f'(z)\left(\frac{z}{f(z)}\right)^2\right]-\arg\left(\frac{z}{f(z)}\right).$$

Using (3.6) and (3.8), we obtain

$$\begin{aligned}\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| &\le\left|\arg\left[f'(z)\left(\frac{z}{f(z)}\right)^2\right]\right|+\left|\arg\left(\frac{z}{f(z)}\right)\right|\\ &<\arcsin(\lambda)+\arcsin(|a_2|+\lambda).\end{aligned}$$

Now, the desired conclusion follows if

$$\arcsin(\lambda)+\arcsin(|a_2|+\lambda)\le\frac{\alpha\pi}{2}.$$

It is easy to verify that we can apply the result: $\arcsin(x)+\arcsin(y)=\arcsin(x\sqrt{1-y^2}+y\sqrt{1-x^2}), x,y\in[-1,1]$ and $x^2+y^2\le 1$. Therefore, $f\in\mathcal{S}_\alpha$, whenever

$$\arcsin\left[\lambda\sqrt{1-(|a_2|+\lambda)^2}+(|a_2|+\lambda)\sqrt{1-\lambda^2}\right]\le\frac{\alpha\pi}{2}.$$

Equivalently,
$$\lambda\sqrt{1-(|a_2|+\lambda)^2} \leq \sin\frac{\alpha\pi}{2} - (|a_2|+\lambda)\sqrt{1-\lambda^2}$$
Squaring both sides of above inequality and then simplifying the resulting terms, we see that it is equivalent to

$$\left[\sqrt{1-\lambda^2}\sin\frac{\alpha\pi}{2} - (|a_2|+\lambda)\right]^2 \geq \lambda^2\cos^2\frac{\alpha\pi}{2}$$

or,

$$\sqrt{1-\lambda^2}\sin\frac{\alpha\pi}{2} - (|a_2|+\lambda) \geq \lambda\cos\frac{\alpha\pi}{2}$$
A simple calculation shows that above inequality is equivalent to
$$2\lambda^2 + 2|a_2|\lambda - \frac{\sin^2(\frac{\alpha\pi}{2}) - |a_2|^2}{1+\cos(\frac{\alpha\pi}{2})} \leq 0,$$
which holds true for $0 < \lambda \leq \lambda_*(\alpha, |a_2|)$, where $\lambda_*(\alpha, |a_2|)$ is given by (3.1). □

For $\alpha = 1$, Theorem 3.1 immediately implies the following result.

Corollary 3.2. *If $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ be in $\mathcal{N}(\lambda)$ and $|a_2| = |f''(0)|/2 \leq 1$, then $f \in \mathcal{S}^*$ whenever $0 < \lambda \leq \frac{-|a_2|+\sqrt{2-|a_2|^2}}{2}$.*

Remark 3.3. In Theorem 3.1, if we replace λ_* by $\lambda/2$, we obtain the following result of Obradović and Ponnusamy [4, Corollary 1.10] :

If $0 < \lambda \leq (-|f''(0)|\cos(\pi\alpha/4) + \sin(\pi\alpha/4)\sqrt{16\cos^2(\pi\alpha/4) - |f''(0)|^2})/2\cos(\pi\alpha/4)$ with $\alpha \in (0, 1]$, then we have the inclusion $\mathcal{P}(\lambda) \subseteq \mathcal{S}_\alpha$.

Theorem 3.4. *Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ be in $\mathcal{N}(\lambda)$ with $|a_2| = |f''(0)|/2 \leq 1$, $\gamma \in (0, 1]$ and λ_γ satisfy the inequality*
$$\sqrt{1-\lambda^2}\sin\gamma\pi/2 \geq 2(|a_2|+\lambda)\sqrt{1-(|a_2|+\lambda)^2} + \lambda\cos\gamma\pi/2. \tag{3.9}$$
Then $f \in R(\gamma)$ for $0 < \lambda \leq \lambda_\gamma$.

Proof. Since
$$|\arg f'(z)| \leq \left|\arg f'(z)\left(\frac{z}{f(z)}\right)^2\right| + 2\left|\arg\left(\frac{z}{f(z)}\right)\right|,$$
Using (3.6) and (3.8) in above inequality, it follows that
$$|\arg f'(z)| < \arcsin(\lambda) + 2\arcsin(|a_2|+\lambda). \tag{3.10}$$

Then from (3.10), we see that $|\arg f'(z)| < \gamma\pi/2$ holds in E whenever

$$\arcsin(\lambda) + 2\arcsin(|a_2| + \lambda) \leq \gamma\pi/2. \tag{3.11}$$

Using the formula $2\arcsin(x) = \arcsin(2x\sqrt{1-x^2}), x \in (-1/\sqrt{2}, 1/\sqrt{2})$ and then by a computation and simplification, we find that inequality (3.11) is equivalent to (3.9). □

If $a_2 = 0$ and $\gamma = 1$, then Theorem 3.4 gives :

Corollary 3.5. *If $f(z) = z + a_3z^3 + a_4z^4 \cdots$ be in $\mathcal{N}(\lambda)$ and $0 < \lambda \leq 1/2$, then $Ref'(z) > 0$ in E.*

Remark 3.6. In Theorem 3.4, if we replace λ by $\lambda/2$, we obtain the following result of Obradović and Ponnusamy [4, Corollary 1.13] :

Let $f \in \mathcal{P}(\lambda)$. Suppose $0 < \lambda \leq 2$ and $0 < \gamma \leq 1$ are given by $\sin(\pi\gamma/2)\sqrt{4-\lambda^2} \geq (|f''(0)| + \lambda)\sqrt{4 - (|f''(0)| + \lambda)^2} + \lambda\cos(\pi\gamma/2)$. Then we have $\mathcal{P}(\lambda) \subset R(\gamma)$.

In the next result, we find conditions under which functions in $\mathcal{N}(\lambda)$ belong to $\mathcal{S}^*(\gamma), 0 \leq \gamma < 1$.

Theorem 3.7. *If $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ be in $\mathcal{N}(\lambda)$ and $|a_2| = |f''(0)|/2 \leq 1$, then $f \in \mathcal{S}^*(\gamma), 0 \leq \gamma < 1$ whenever $0 < \lambda \leq \frac{(1-\gamma)-(\gamma+1)|a_2|}{\sqrt{2}+\gamma}$.*

Proof. As $f \in \mathcal{N}(\lambda)$. Then, from (3.5) and in view of (3.3) and (3.7), we obtain

$$\frac{zf'(z)}{f(z)} = \frac{f'(z)\left(\frac{z}{f(z)}\right)^2}{\frac{z}{f(z)}} = \frac{1 + \lambda\int_0^1 \frac{w(tz)}{t^2}\log(1/t)dt}{1 - a_2z - \frac{\lambda}{2}\int_0^1 \frac{w(tz)}{t^2}(\log(1/t))^2dt}, \quad w \in \mathcal{B}_1 \text{ and } z \in E. \tag{3.12}$$

Now $f \in \mathcal{S}^*(\gamma)$ is equivalent to the condition:

$$\frac{1}{1-\gamma}\left(\frac{zf'(z)}{f(z)} - \gamma\right) \neq iT, \text{ for all } T \in \mathbb{R} \text{ and } z \in E. \tag{3.13}$$

Using (3.12) in (3.13), we get

$$\lambda\int_0^1 \frac{w(tz)}{t^2}\log(1/t)dt - \gamma(1-iT)\left[1 - a_2z - \frac{\lambda}{2}\int_0^1 \frac{w(tz)}{t^2}(\log(1/t))^2dt\right]$$
$$-iT\left[-a_2z - \frac{\lambda}{2}\int_0^1 \frac{w(tz)}{t^2}(\log(1/t))^2dt\right] \neq -(1-iT),$$

or equivalently,

$$\begin{aligned}
&\frac{1}{2}\left[\lambda\int_0^1\frac{w(tz)}{t^2}\log(1/t)dt+\left(-a_2z-\frac{\lambda}{2}\int_0^1\frac{w(tz)}{t^2}(\log(1/t))^2dt\right)\right]\\
&+\frac{1}{2}\frac{1+iT}{1-iT}\left[\lambda\int_0^1\frac{w(tz)}{t^2}\log(1/t)dt-\left(-a_2z-\frac{\lambda}{2}\int_0^1\frac{w(tz)}{t^2}(\log(1/t))^2dt\right)\right]\\
&-\gamma\left[1-a_2z-\frac{\lambda}{2}\int_0^1\frac{w(tz)}{t^2}(\log(1/t))^2dt\right]\neq -1. \qquad (3.14)
\end{aligned}$$

If we denote the left-hand side of (3.14) by $H(w,T,z)$ and let

$$M=\sup_{T\in\mathbb{R},w\in\mathcal{B}_1,z\in E}|H(w,T,z)|$$

then, in view of the rotation invariance property of the space $\mathcal{B}_1$, we see that $f\in\mathcal{S}^*(\gamma)$ if $M\leq 1$. It can be seen that

$$\begin{aligned}
M\leq\sup_{w\in\mathcal{B}_1,z\in E}&\frac{\lambda}{2}\left|\int_0^1\frac{w(tz)}{t^2}\log(1/t)dt-\frac{1}{2}\int_0^1\frac{w(tz)}{t^2}(\log(1/t))^2dt\right|\\
&+\frac{\lambda}{2}\left|\int_0^1\frac{w(tz)}{t^2}\log(1/t)dt+\frac{1}{2}\int_0^1\frac{w(tz)}{t^2}(\log(1/t))^2dt\right|+|a_2||z|\\
&+\gamma\left|1-|a_2||z|-\frac{\lambda}{2}\int_0^1\frac{w(tz)}{t^2}(\log(1/t))^2dt\right|
\end{aligned}$$

Using the Parallelogram Law: $|z_1+z_2|+|z_1-z_2|\leq 2\sqrt{|z_1|^2+|z_2|^2}$, we have

$$\begin{aligned}
M\leq\lambda\sup_{w\in\mathcal{B}_1,z\in E}&\left[\sqrt{\left|\int_0^1\frac{w(tz)}{t^2}\log(1/t)dt\right|^2+\left|\frac{1}{2}\int_0^1\frac{w(tz)}{t^2}(\log(1/t))^2dt\right|^2}\right]\\
&+\gamma+(1+\gamma)|a_2||z|+\gamma\frac{\lambda}{2}\left|\int_0^1\frac{w(tz)}{t^2}(\log(1/t))^2dt\right|.
\end{aligned}$$

As $w\in\mathcal{B}_1$, so using the fact $|w(z)|\leq|z|^2$ in E, we get

$$\begin{aligned}
M&\leq\lambda\sup_{z\in E}\sqrt{\left(\int_0^1\log(1/t)dt\right)^2+\left(\int_0^1\frac{1}{2}(\log(1/t))^2dt\right)^2}\,|z|^2\\
&\quad+\gamma+(1+\gamma)|a_2||z|+\gamma\frac{\lambda}{2}\left(\int_0^1(\log(1/t))^2dt\right)|z|^2,\\
&=\lambda(\sqrt{2}+\gamma)|z|^2+\gamma+(1+\gamma)|a_2||z|
\end{aligned}$$

Thus (3.13) (or 3.14) holds if $0<\lambda\leq\frac{(1-\gamma)-(\gamma+1)|a_2|}{\sqrt{2}+\gamma}$. $\square$

Corollary 3.8. *Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ be in $\mathcal{N}(\lambda)$ and $a_2 = f''(0)/2 = 0$. Then $f \in \mathcal{S}^*(\gamma), 0 \le \gamma < 1$ provided $0 < \lambda \le \frac{1-\gamma}{\sqrt{2}+\gamma}$.*

Choosing $\gamma = 0$ and $a_2 = f''(0)/2 = 0$ in Theorem 3.7, we have the following

Corollary 3.9. *Let $f(z) = z + a_3 z^3 + a_4 z^4 \cdots$ be in $\mathcal{N}(\lambda)$ then $f \in \mathcal{S}^*$ provided $0 < \lambda \le \frac{1}{\sqrt{2}}$.*

Theorem 3.10. *Let $f \in \mathcal{U}$ be of the form $z/f(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ and $g(z) = \frac{1}{r} f(rz), z \in E$. Then $g \in \mathcal{N}$ for $0 < r \le r_0$, where $r_0 \approx 0.4913$ is the unique positive root of the equation*

$$2r^{10} + 6r^8 + 21r^6 - 9r^4 + 5r^2 - 1 = 0. \tag{3.15}$$

Proof. Since $f \in \mathcal{U}$ and has the form

$$f(z) = \frac{z}{1 + \sum_{n=1}^{\infty} b_n z^n}, \ z \in E, \tag{3.16}$$

Therefore, by Lemma 2.1, we have

$$\sum_{n=2}^{\infty} (n-1)^2 |b_n|^2 \le 1. \tag{3.17}$$

Using (3.16), for $0 < r \le 1$, we can write

$$\frac{z}{\frac{1}{r} f(rz)} = 1 + \sum_{n=1}^{\infty} (b_n r^n) z^n.$$

In view of Lemma 2.2, we need to show

$$\sum_{n=2}^{\infty} (n-1)^3 |b_n r^n| \le 1, \ \text{for } \ 0 < r \le r_0.$$

Now, in view of Cauchy-Schwarz inequality and (3.17), we have

$$\begin{aligned} \sum_{n=2}^{\infty} (n-1)^3 |b_n| r^n &\le \left(\sum_{n=2}^{\infty} (n-1)^2 |b_n|^2 \right)^{1/2} \left(\sum_{n=2}^{\infty} (n-1)^4 r^{2n} \right)^{1/2} \\ &\le \left(\sum_{n=2}^{\infty} (n-1)^4 r^{2n} \right)^{1/2} \\ &= \left(\frac{r^4 (1 + 11r^2 + 11r^4 + r^6)}{(1-r^2)^5} \right)^{1/2} \le 1, \end{aligned}$$

provided $0 < r \le r_0$. Using Mathematica, One can verify that $\sum_{n=2}^{\infty} (n-1)^4 r^{2n} = \frac{r^4(1+11r^2+11r^4+r^6)}{(1-r^2)^5}$ and that equation (3.15) has only one positive real root in $(0, 1)$. □

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Department of Mathematics, Sant Longowal Institute of Engineering and Technology, Longowal-148106, Punjab, India

E-mail address: m.ahluwalia10@gmail.com [1], sushmagupta1@yahoo.com [2] , sukhjit_d@yahoo.com [3]